Dual of $\mathcal{C}_0(X)$.

Recall: (three • below)

Banach space

- The dual X^* of a normed space $(X, \|\cdot\|)$ is the (complete) vector space of bounded linear functionals ($\Leftrightarrow$ continuous linear functionals) $X \to \mathbb{C}$.

- $\mathcal{C}_c(X)$ is a normed space under the uniform norm $\|\varphi\|_u = \sup\{|\varphi(x)| : x \in X\}$. $(\mathcal{C}_c(X), \|\cdot\|_u)$ is not complete and its completion is $\mathcal{C}_0(X)$. Since convergence in $\|\cdot\|_u \Rightarrow$ uniform convergence on compact subsets, standard analysis $\Rightarrow \mathcal{C}_0(X) \subseteq \mathcal{C}(X)$.

<u>Note:</u> $f \in C_0(X) \Rightarrow \forall \varepsilon > 0\ \exists$ cpct $K \subseteq X$ s.t. $|f(x)| < \varepsilon$ in $X \setminus K$.

- Riesz Repr. Thm. (Basic) version that we proved identifies every positive linear functional I on $C_c(X)$ w/ a Radon measure μ s.t.

$$I(\varphi) = \int \varphi \, d\mu,$$

where

all open U.

(1) $\mu(U) = \sup\{I(\varphi) : \varphi \in C_c(X),\ \varphi \prec U\}$,

(2) $\mu(K) = \inf\{I(\varphi) : \varphi \in C_c(X),\ \varphi \geq \chi_K\}$,

all cpct K.

<u>Rem.</u> The functionals in RRT above need not be bdd (i.e. $|I(\varphi)| \leq C\|\varphi\|_u$), but do need to be positive.

It is clear that a bdd linear $I \in \mathcal{C}_0(X)^*$ restricts to a bdd linear functional on $\mathcal{C}_c(X)$. Conversely, if $I \in \mathcal{C}_c(X)^*$, then it extends as an $I \in \mathcal{C}_0(X)^*$ by continuity: Take $f \in \mathcal{C}_0(X)$ and let $\varphi_n \in \mathcal{C}_c(X)$ s.t. $\|f - \varphi_n\|_u \to 0$. Then, define $I \in \mathcal{C}_c(X)$ by $I(f) = \lim_{n\to\infty} I(\varphi_n)$. Easy to check that lim exists and is indep of approximating seg. $\{\varphi_n\}$.

Rem. Since we will be discussing Radon measures, X will be LCH space until further notice.

Lemma 1. Let μ be Radon measure on X. The linear functional (positive)

$$I(\varphi) = \int \varphi \, d\mu, \quad \varphi \in C_c(X)$$

is bdd (extends to $C_0(X)^*$) $\Leftrightarrow$ $\mu(X) < \infty$.

Pf. $\Leftarrow$ is clear:

$$|I(\varphi)| \leq \int |\varphi| \, d\mu \leq \mu(X) \|\varphi\|_u.$$

by (1)

$\Rightarrow$: $\mu(X) = \sup\{\int \varphi \, d\mu : \varphi \in C_c(X), 0 \leq \varphi \leq 1\}$

Since I is assumed bdd $\Rightarrow$ $|\int \varphi \, d\mu| \leq C \|\varphi\|_u$

For $0 \leq \varphi \leq 1$, $|\int \varphi \, d\mu| = \int \varphi \, d\mu \Rightarrow$

$\mu(X) \leq \sup\{C \|\varphi\|_u : \varphi \in C_c(X), 0 \leq \varphi \leq 1\}$

$= C.$ ∎

Complex Radon measures (X LCH space).

Recall: • A cplx Borel measure μ is a function $\mathcal{B}_X \to \mathbb{C}$ (every Borel set, including X, takes a cplx value) that can be decomposed $\mu = \mu_1 + i\mu_2$, where μ_1, μ_2 are signed measures taking only finite values. The signed measures μ_j can further be decomposed as $\mu_j = \mu_j^+ - \mu_j^-$, where $\mu_j^+ \perp \mu_j^-$ (Jordan Decomp. Thm).

• Total variation $|\mu|$: For a signed measure ν, $|\nu| = \nu^+ + \nu^-$. For a cplx measure μ, let $\mu = \nu_1 + i\nu_2$, $\gamma = |\nu_1| + |\nu_2|$, pos., finite and $d\mu = f\, d\gamma$ (Lebesgue-Radon-Nikodym since $\mu \ll \gamma$)

$L^1(X, \gamma)$

Then, $d|\mu| = |f|\,dy$.

<u>Def.</u> A cplx measure $\mu = \mu_1 + i\mu_2$, $\mu_j = \mu_j^+ - \mu_j^-$, is Radon if the measures $\mu_j^\pm$ are all Radon.

<u>Prop1.</u> ① A cplx measure μ is Radon $\Leftrightarrow$ $|\mu|$ is Radon.

② The space $M(X)$ of cplx Radon measures is a normed vector space with

$$\|\mu\| = |\mu|(X).$$

<u>Pf.</u> See Prop 7.16 in Folland.

<u>Riesz Rep. Thm.</u> Let X be LCH space, and, for $\mu \in M(X)$ and $f \in C_0(X)$, let $I_\mu(f) = \int f\,d\mu$. The map $\mu \to I_\mu$ is an isometric isomorphism $M(X) \to C_0(X)^*$.

<u>Rem.</u> If X is cpct, then $\mathcal{C}_0(X) = \mathcal{C}_c(X) = \mathcal{C}(X)$. Thus, an immediate cor. of RRT is

<u>RRT'</u>. Let X be cpct Hausdorff space. Then, $\mu \to I_\mu$ is isometric isomorphism $M(X) \to \mathcal{C}(X)^*$.

For pf of RRT, we need:

<u>Lemma 2</u>. If $I \in \mathcal{C}_0(X,\mathbb{R})^*$, then $\exists$ positive $I^+, I^- \in \mathcal{C}_0(X,\mathbb{R})^*$ s.t. $I = I^+ - I^-$.

<u>Pf</u>. Decompose $f \in \mathcal{C}_0(X,\mathbb{R})$ as $f = f^+ - f^-$, where $f^+(x) = \max(f(x), 0)$, $f^-(x) = \max(-f(x), 0)$, so $f^\pm \geq 0$.

Define I^+ on $f \geq 0$ by

$$I^+(f) = \sup\{I(g) : g \in C_0(X,\mathbb{R}),\ 0 \leq g \leq f\}$$

Since $g=0$ is a candidate in sup and $I(0)=0$, $I^+(f) \geq 0$. Also, $I \in C_0(X,\mathbb{R})^*$

Banach space w/ operator norm $\|\cdot\|$

$\Rightarrow$

$$I^+(f) = \sup\{\max(0, I(g)) : \text{as above}\}$$

and if $I(g) \geq 0$, then

$$I(g) = |I(g)| \leq \|I\|\,\|g\|_u$$

$$\Rightarrow \quad 0 \leq I^+(f) \leq \|I\|\,\|f\|_u \quad (3)$$

For general $f \in C_0(X,\mathbb{R})$, let

$$I^+(f) = I^+(f^+) - I^+(f^-).$$

I^+ is clearly positive ($I^+(f) \geq 0$ if $f \geq 0$).
To check that $I^+ \in C_0(X)^*$, suffices to check:
For $f, f_1, f_2 \geqslant 0$, $c \geqslant 0$:

(a) $I^+(cf) = cI^+(f)$ (pretty obvious) ✓

(b) $I^+(f_1+f_2) = I^+(f_1) + I^+(f_2)$

(c) $\underset{\geq 0 \text{ by pos.}}{I^+(f)} \leq \|I\| \|f\|_u$. (already proved: (3))

<u>Let's prove (b)</u>: If $0 \leq g_1 \leq f_1$, $0 \leq g_2 \leq f_2$, then $0 \leq g_1 + g_2 \leq f_1 + f_2 \Rightarrow I^+(f_1+f_2) \geq I(g_1) + I(g_2)$
$\Rightarrow I(f_1+f_2) \geq I^+(f_1) + I^+(f_2)$.
If $0 \leq g \leq f_1 + f_2$, then setting $g_1 = \min(g, f_1)$
$\Rightarrow 0 \leq g_1 \leq f_1$. W/ $g_2 = g - g_1$ we get
$0 \leq g_2 \leq f_2$ (consider 2 cases $g_1(x) = g(x)$ or $g_1(x) = f_1(x)$)

Pick $\varepsilon > 0$. $\exists\, 0 \le g \le f_1 + f_2$ s.t.

$$I^+(f_1+f_2) \le I(g) + \varepsilon = I(g_1) + I(g_2) + \varepsilon$$

$$\le I^+(f_1) + I^+(f_2) + \varepsilon \Rightarrow \{\varepsilon \to 0\}$$

$$I^+(f_1+f_2) \le I(f_1) + I(f_2) \Rightarrow \{\text{previous ineq.}\}$$

$$I^+(f_1+f_2) = I(f_1) + I(f_2), \text{ i.e. (b)}$$

Thus, we have established $I^+ \in C_0(X,\mathbb{R})^*$.

Set $I^- = I^+ - I \in C_0(X,\mathbb{R})$. To complete pf of Lemma 2, just need to verify that I^- is positive, or $I^+(f) \ge I(f)$ for $f \ge 0$. This is obvious, since f itself is a candidate in the sup that defines $I^+(f)$. ☒

<u>Pf of RRT.</u> Consider the map $I_\mu(f) = \int f\,d\mu$, $f \in C_0(X)$. It is clearly a linear functional and

$$|I_\mu(f)| \leq \int |f|\,d|\mu| \leq |\mu|(X)\,\|f\|_u$$

$\Rightarrow$ $I_\mu \in C_0(X)^*$ and $\|I_\mu\| \leq \|\mu\|$.

The map $I_\mu : M(X) \to C_0(X)^*$ is clearly linear $/\mathbb{C}$ and injective. Need to establish:

① I_μ is surjective. ($\Rightarrow$ isomorphism)

② $\|I_\mu\| = \|\mu\|$. (Already have $\|I_\mu\| \leq \|\mu\|$ so suffices to show $\|I_\mu\| \geq \|\mu\|$)

For ①, pick $I \in C_0(X)^*$, decompose $I = I_1 + i I_2$, $I_j \in C_0(X,\mathbb{R})$ and use Lemma 2, to decompose $I_j = I_j^+ - I_j^-$, where

$I_j^+, I_j^- \in \mathcal{C}_0(X,\mathbb{R})^*$ are positive.
By RRT (basic) and continuity of $I_j^\pm$, we conclude that $\exists$ Radon measures $\mu_j^\pm$ s.t.

$$I_j^\pm(f) = \int f \, d\mu_j^\pm, \quad f \in \mathcal{C}_0(X)$$

and

$$\mu_j^\pm(X) = \sup\{I_j^\pm(g) : g \in \mathcal{C}_c(X), 0 \le g \le 1\}$$
$$\le \|I_j^\pm\|.$$

Reassembling I from $I_j^\pm$, we find that the complex Radon measure $\mu = (\mu_1^+ - \mu_1^-) + i(\mu_2^+ - \mu_2^-)$ satisfies $I_\mu = I$.
This proves 1 (surjectivity of $\mu \to I_\mu$).

To prove 2, we must demonstrate

$$\|I_\mu\| \ge \|\mu\|.$$

Since $\mu \ll |\mu| \Rightarrow d\mu = h\, d|\mu|$, where $|h| = 1$ a.e. $(|\mu|)$ (see Prop 3.13 (b)) $\Rightarrow$

$$\|\mu\| = |\mu|(X) = \int |h|^2 d|\mu| = \int \bar{h} \cdot h\, d|\mu|$$
$$= \int \bar{h}\, d\mu.$$

Pick $\varepsilon > 0$. By Lusin $\exists\, \varphi \in C_c(X)$ and $E \subseteq X$ s.t. $|\mu|(E) < \varepsilon$ and $\varphi = \bar{h}$ on $X \setminus E$, $\|\varphi\|_u \leq \|h\|_u = 1$

$$\|\mu\| = \int \bar{h}\, d\mu = \int \varphi\, d\mu + \int (\bar{h} - \varphi)\, d\mu$$

$$\Rightarrow \|\mu\| \leq \left| \int \varphi\, d\mu \right| + 2|\mu|(E) < |I_\mu(\varphi)| + 2\varepsilon$$
$$\leq \|I_\mu\| + 2\varepsilon \Rightarrow \{\varepsilon \to 0\} \Rightarrow$$

$\|\mu\| \leq \|I_\mu\|$ as desired. This completes pf of RRT. $\blacksquare$